Approximate Computation of Underexpanded Jet Structure

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Introduction

AN investigation of the structure of a jet, which exhausts at a pressure much greater than the ambient pressure has been carried out by an approximate analysis. This analysis has application in the description of the structure of a rocket exhaust at ionospheric altitudes and of laboratory jet facilities of extreme pressure ratio. The effort presented here is essentially the first step in a logical sequence of further refined analyses. In this work, continuum fluid descriptors and equations will be used, along with several simplifying assumptions. These are 1) neglect of viscous and diffusion effects, 2) neglect of magnetic field effect on pressure distribution, 3) use of Newtonian thin shock-layer approximation, and 4) neglect of atmospheric density gradient.

The justification for each of these simplifications is as follows. The effects of finite transport coefficients are limited to influencing the details of velocity profiles and molecular species distributions. These are moderate insofar as the description of pressure and density fields are concerned, and this latter description is the basic effort here. The pressure and density fields in the core of the jet, i.e., the region interior to the jet boundary shock, will be affected negligibly by finite transport coefficients. From experience with results for viscous flows about solid bodies, it seems reasonable that the influence of these effects on the shock-layer pressure and density patterns will be moderate at most. The fluid field descriptions can be modified a posteriori by a perturbation calculation, which includes the results of this work as a starting or unperturbed condition.

At F-layer altitudes, the atmosphere[1] is ionized to the extent of $\sim 0.1\%$, and the magnetic field strength is of the order of $\frac{1}{2}$ gauss. This gives rise to a magnetic pressure on the neutral plus ionized gas mixture of $\sim 10^{-11}$ atm, so that the influence of the ions on the flow field is negligible. The converse is, of course, not true, particularly in that region of the flow far downstream of the nozzle exit plane where the jet cavity begins to be filled by radial flow from the atmosphere. In this zone the ions will presumably pursue a course of motion dominated by the geomagnetic field. We treat the fluid as neutral in order to determine a pressure field, which may be used later to examine the ion and electron motion in this environment.

The use of the Newtonian fluid flow in the present description is justified to the extent that the order of accuracy of this approximation at the Mach numbers of interest ($M_\infty \sim 3$ to 5) is well established[2] and is consistent with the degree of precision intended here.

The neglect of atmospheric density gradient is also consistent with this order of precision. The severity of this approximation may be estimated by computing the fractional density change over a characteristic length. As will be demonstrated, the relevant characteristic length is the square root of the jet thrust divided by the ambient static pressure, which, in regions of interest here, is of the order of tens of kilometers for thrusts $\gtrsim 10^4$ lbf. Since an atmospheric scale height is of the order of 10 km at ionospheric altitudes,[1] the error due to neglect of density variation is seen to be of the order of 10%, here an acceptable value. Note that the larger the thrust, the greater is the influence of the atmospheric gradient.

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Statement of Problem

Consider a jet exhausting axially into a moving stream of air, the jet flow and the air flow being aligned. The gross features of the jet flow will be unaffected by the presence of the ambient stream until some considerable distance from the nozzle, where the jet density has fallen to such value that the jet dynamic pressure is comparable to the local airstream impact pressure. Only the boundary zone of the jet need be described in order to complete the picture. Under the forementioned approximations, one must describe the boundary zone of the interaction of the asymptotic jet flow in vacuum with the ambient airstream. Let r be the radial distance from the exit and θ the polar angle measured from the jet centerline. The azimuthal angle is irrelevant by symmetry. The radius of curvature of the streamline dividing internal and external flow is R_c, and the coordinate system is fixed to the jet.

In this coordinate system, the zero-pressure limit of the Euler equations gives for jet flow density distribution:

$$\rho = r^{-2}f(\theta) \qquad f \text{ is any function} \tag{1}$$

whereas the velocity is radial and everywhere constant, equal to $C_{\max}$, the limiting speed.

A simple approximation for the Mach number angular distribution has been employed which seems to approximate exact computation adequately.[3] This approximation is

$$\lim_{r\to\infty}\left[\frac{M(\theta = 0)}{M(\theta)}\right] \doteq \cos^{1/2}\left(\frac{\pi}{2}\frac{\theta}{\theta_m}\right) \tag{2}$$

where θ_m is the expansion angle to vacuum.

In the vacuum expansion case, the characteristic length is the throat radius of the jet (sonic condition dimension). In the boundary region of interest here, the distance in units of this dimension is essentially infinity, so that only the preceding asymptotic results need be employed in the jet description. The fractional mass flow parameter $g(\theta)$ can be written as

$$g(\theta) = \frac{m_j(\theta)}{m_j(0)} = \frac{\int_\theta^{\theta_m} \cos^{1/\gamma - 1}\left(\frac{\pi}{2}\frac{\theta'}{\theta_m}\right)\sin\theta'\, d\theta'}{\int_0^{\theta_m} \cos^{1/\gamma - 1}\left(\frac{\pi}{2}\frac{\theta'}{\theta_m}\right)\sin\theta' d\theta'} \tag{3}$$

where $m_j(\theta)$ is the mass flow in the jet outside of a cone of interior half-angle θ.

Equation for Dividing Streamline

Under the Newtonian thin shock-layer assumption and using a homogeneous shock-layer model, the pressure at the dividing streamline (including centrifugal relief) caused by the ambient airstream is equated to the pressure computed from the jet core outward. This gives the scaling mentioned previously.† Using the length l^*

$$l^* = [(F/P_\infty)(C_{\max}/C^*)(1/2\pi)]^{1/2} \tag{4}$$

where F is jet thrust and C^* the jet characteristic velocity, and denoting by asterisk the dimensionless distances, we find that

$$R_c^* = r^*\left[\frac{r^{*2}(\gamma_\infty M_\infty{}^2/2)\sin^2\theta\cos\phi + g(\theta)\cos(\theta - \phi)}{r^{*2}\sin\theta\,(1 + \gamma_\infty M_\infty{}^2\sin^2\phi) + g'(\theta)\sin^2(\theta - \phi)}\right] \tag{5}$$

where θ is the inclination of the boundary to freestream velocity.

This equation links the radius of curvature to r, θ, and φ; two additional geometrical relations are also available by vir-

† A similar scaling law has been found by Alden,[3] Lam,[7] and surely by others. For an excellent bibliography and detailed exposition, see Rosner.[8]

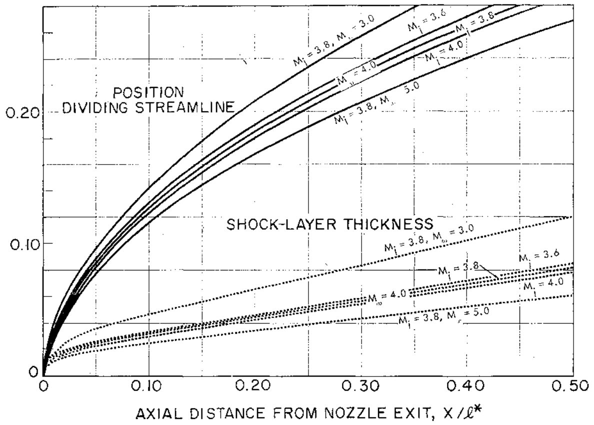

Fig. 1 Effect of jet and airstream Mach number variations on shock-layer geometry.

tue of the thin shock-layer approximation, namely,

$$R^* \sin(\theta - \phi) = r^*(d\theta/d\phi) \tag{6}$$

$$r^* \operatorname{ctn}(\theta - \phi) = -(dr^*/d\theta) \tag{7}$$

Once initial values of r^*, θ, φ are specified, Eqs. (5–7) can be integrated stepwise to provide an arc $r^*(\theta)$ that describes the dividing streamline.

Initial Conditions

In all of the cases of interest (jet exit pressure/ambient pressure $>10^3$), the initial polar angle θ_m' of the radial flow will equal or exceed the maximum flow-deflection angle for an attached bow shock, giving rise to a detached wave standing ahead of the exhaust plume. In many of the cases, θ_m' will exceed 90°. If we presume also that the initial bow-shock shape is parabolic, the initial radius of curvature of the bow shock R_c^0 is maintained for some distance. We suppose that this approximation is valid back to some distance at which the bow-shock layer is thin. In order to define the point (r_1, θ_1) at which this occurs, we make the following argument.

The drag of a body of revolution which has the shape of the dividing streamline must equal the difference in axial momentum flux between the exhaust flow in the inner shock layer and the same portion of the exhaust flow when the jet exhausts into a vacuum. To find the drag, we postulate zero base pressure $C_p = 2$ and use a body terminated at the point where the bow-shock layer becomes thin (at r_1, θ_1, φ_1). Equating the momentum flux difference to the drag gives, using the dimensional parameter of Eq. (4),

$$g(\theta_1) \cos\phi_1 \cos(\theta_1 - \phi_1) - g(\theta_1)\cos\theta_1 + \int_{\theta_1}^{\theta_m} g(\theta) \sin\theta d\theta = \frac{1}{2}(1 + \gamma_\infty M_\infty^2)(r_1^* \sin\theta_1)^2 \tag{8}$$

This equation, in conjunction with Eq. (5) and an assumed value for θ_1, gives two simultaneous equations for three unknowns. From symmetry it is required that the radius of curvature R_c^0 terminate on the jet centerline, so that

$$R_c^{0*} \cos\phi_1 = r_1^* \sin\theta_1 \tag{9}$$

The numerical integrations were carried out by implementing Gaussian quadrature[4] and harmonic analysis[5] procedures on a digital computer. Such procedures exemplify the advantages that can accrue to approximate analysis by the judicious choice of approximate numerical procedures and digital computer application.‡

‡ More explicit development and a copy of the code may be found in Ref. 6.

Conditions in Shock Layer

To estimate conditions in the shock layer, the following were computed: 1) conditions on the stagnation-point streamline, 2) conditions just behind the bow shock, and 3) shock-layer thickness. Consistent with the present inviscid-flow approximation, conditions on the stagnation streamline are determined by isentropic expansion from stagnation conditions, and conditions just behind the shock are given by the Hugoniot equation. These two density ratios then serve as approximate bounds to the density variations in the bow-shock layer.

The thickness of the bow-shock layer (t) was estimated by the following procedure. When the shock layer is thin, the mass conservation equation can be written as

$$\frac{d(r\sin\theta)}{dt} = 2\frac{\rho_b}{\rho_\infty}\cos\phi = 2\frac{\gamma_\infty + 1}{\gamma_\infty - 1}\cos\phi\left[1 + \frac{2}{(\gamma_\infty - 1) M_\infty^2 \sin^2\phi}\right]^{-1} \tag{10}$$

Using Eq. (7), then, gives $dt^*/d\theta$ in the form

$$\frac{dt^*}{d\theta} = \frac{r^*}{2}\frac{\gamma_\infty - 1}{\gamma_\infty + 1}\left[1 + \frac{2}{(\gamma_\infty - 1) M_\infty^2 \sin^2\phi}\right] \times \left[\frac{\cos\theta - \sin\theta \operatorname{ctn}(\theta - \phi)}{\cos\phi}\right] \tag{11}$$

This expression (with t^* and r^* made dimensionless by l^*) was integrated stepwise and is displayed with the results of the example cases. An initial value t_1^* was estimated by using the homogeneous layer model at the initial point (r_1^*, θ_1, φ_1), giving

$$t_1^* \doteq \frac{1}{2}\frac{\gamma_\infty - 1}{\gamma_\infty + 1}\left[1 + \frac{2}{(\gamma_\infty - 1) M_\infty^2 \sin^2\phi_1}\right]\frac{\sin\theta_1}{\cos\phi_1} r_1^* \tag{12}$$

Numerical Results

Figure 1 gives a cross-sectional view of the dividing streamline surface (in dimensionless coordinates) for five flight conditions. A nozzle half-angle of 5° was used, with $\gamma = 1.22$ for the jet gas.

Note that the dividing streamline shapes are well approximated by functions of the form

$$y^* = ax^{*1/2} \ (a \approx 4) \tag{13}$$

Using this representation, one can evaluate the severity of the approximation of the homogeneous shock layer. Carrying out the required manipulations reveals that the bow-shock layer centrifugal-pressure relief is approximated to within about 10% by this model. This accuracy is adequate, and the approximation becomes better with increasing distance from the exit.

Figure 2 presents the approximate results of the shock-layer density calculations for the same sets of flight conditions.

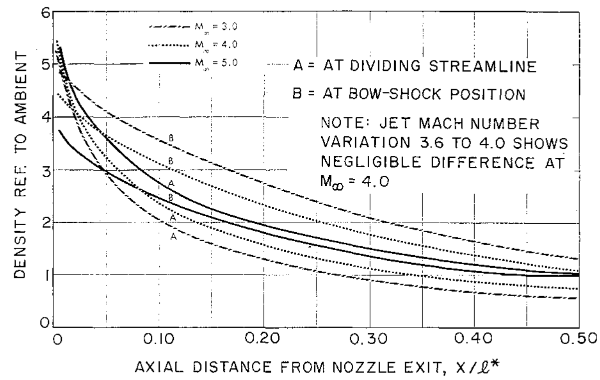

Fig. 2 Effect of airstream Mach number on shock-layer density.

The slight variation of jet exit Mach number produced negligible change in the curves for $M_\infty = 4.0$, as might have been inferred from the similarity of this family of curves in Fig. 1.

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Approximate Inviscid, Nonadiabatic Stagnation Region Flow Field Solution

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Nomenclature

$\bar{\rho}$ = nondimensionalized density, ρ/ρ_i
$\bar{h}$ = nondimensionalized static enthalpy, h/h_i
$\bar{v}$ = nondimensionalized radial velocity component, v/v_i
$\bar{y}$ = nondimensionalized curvilinear coordinate normal to shock front, y/δ
$\bar{\delta}$ = defined as δ/δ_0
$\bar{q}_i$ = defined as $I_i\delta_0/\rho_\infty V_\infty{}^3$
$\bar{q}$ = defined as $2q/\rho_\infty v_\infty{}^3$
δ = shock-detachment distance
q = radiation heat-transfer rate
I = radiation intensity per unit volume
η = $(n - m)$ where n, m are exponents in the intensity, density, enthalpy relationship[4]

Subscripts

i = initial conditions behind normal shock
∞ = freestream conditions
0 = no radiation occurring in the flow

In the study of radiative heat transfer in various planetary atmospheres, it is of considerable interest to obtain approximate analytical solutions to the inviscid, nonadiabatic flow field in the region of the stagnation point in order to obtain rapid estimates of the radiation heat transfer. However, in utilizing the nonadiabatic energy equation in the stagnation region to obtain values of the radiation heat transfer, various forms of the velocity profile in the shock layer have been assumed. For example, in a previous analysis by Hanley et al.,[1] and more recently by Goulard[2] and Chin et al.,[3] the velocity profile $\bar{v}$ was expressed as a linear function of the shock-layer coordinate $\bar{y}$. Improvements to these analyses have been accomplished[4] by assuming a quadratic velocity profile, thereby achieving greater accuracy. However, it has been shown by Hanley and Korkan[4] that, at relatively large values of radiation transport, the shock-layer velocity profile begins to change character as shown in Fig. 1, invalidating the assumed velocity-profile expression at large values of radiation transport. Investigation of the parameter $\bar{\rho}\bar{v}$ indicates that this variable does not demonstrate this behavior and is relatively insensitive to radiation transport as is shown by the IBM 7094 results in Fig. 1. Therefore, the shock-layer profile of $\bar{\rho}\bar{v}$, as generated by the IBM 7094 program, can be described by an empirical equation of the form

$$\bar{\rho}\bar{v} = \tfrac{1}{2}(\bar{y}^2 - 3\bar{y} + 2) \tag{1}$$

for axisymmetric flow and, as shown in Fig. 1, agrees well

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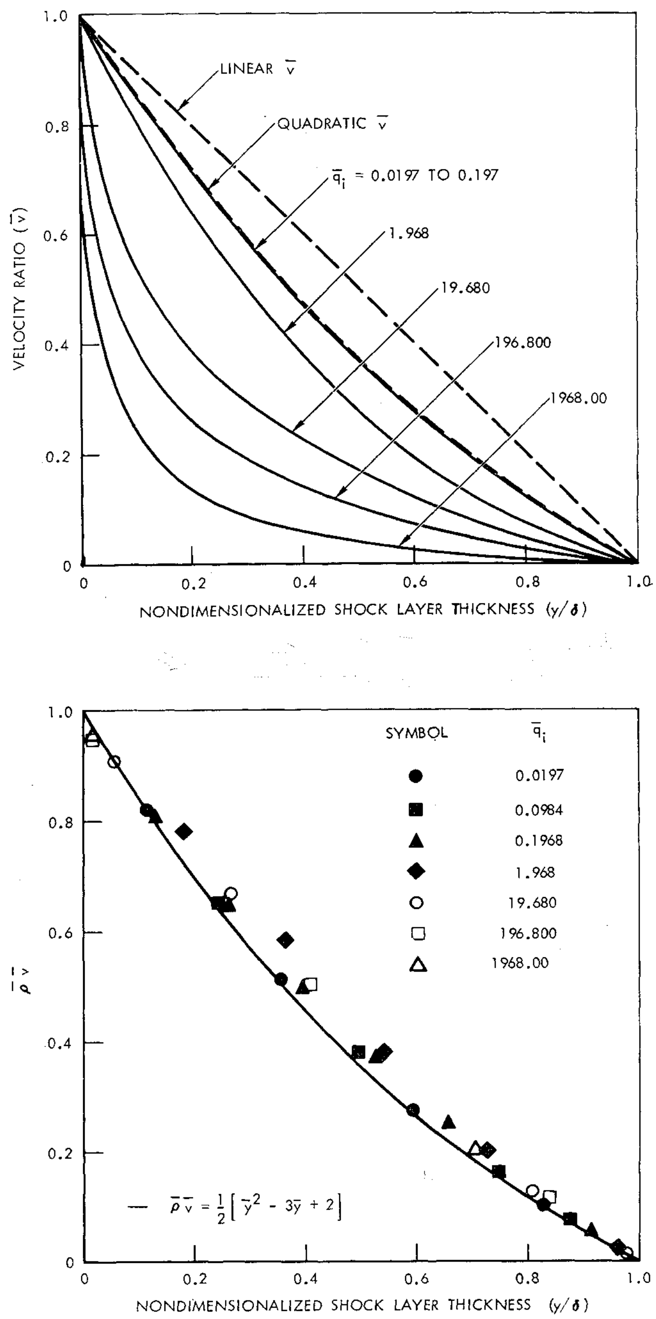

Fig. 1 Variation of velocity profile and parameter $\bar{\rho}\bar{v}$ in shock layer with radiation transport.